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TWO-FLUID MODELING IN ANALYZING THE INTERFACIAL STABILITY OF LIQUID FILM FLOWS

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Abstract—The area-averaged two-fluid model formulation of a separated two-phase flow system is used to investigate interfacial stability of liquid film flows. The analysis takes into account the effects of phase change at the interface as well as the dynamic effects of the adjacent vapor flow on the interfacial stability. Wave formation and instability criteria are established in terms of the generalized fluid and flow parameters. The criteria are applied to investigate the stability of laminar liquid film flow with interfacial shear and phase change. The influence of various dimensionless parameters characterizing film thickness, gravity, phase change and interfacial shear are studied with respect to the neutral stability, temporal growth factor and the wave propagation velocity. The results of the present study indicate that the interfacial stability analysis developed within the frame of the two-fluid model formulation proves to be quite accurate as judged by comparing its results with the available experimental data and with the results of much longer and more complex analytical investigations which are valid only for the liquid film free of interfacial shear.

INTRODUCTION

The flow of liquid films adjacent to a gas or vapor flow is a separated two-phase flow pattern of interest to various technologies because many engineering operations and systems are greatly affected by the behavior of such films. The problem considered in this study is of importance in the chemical process, nuclear reactor and power generating industries. The performance of various units used in these plants is influenced by the dynamics of flow fields because the processes of heat and mass transfer, which occur in these systems, are intimately connected to fluid motion. For example, the increased rates of transport of momentum (Dukler 1972), heat (Chand & Rosson 1965; Williams *et al.* 1968; Fedotkin & Firisyuk 1969), and of mass (Emmert & Pigford 1954; Jepsen *et al.* 1966; Stainthrop & Wild 1967), in both liquid and gas phases, are very often related to the wavy nature of the film flow interface.

For the purpose of providing basic information on the mechanism by which the waves are built-in at the interface, and hence on the mechanisms of associated transport processes in wavy flows, the problem of interfacial stability of liquid films has received considerable attention in the literature. The references dealing with various flow configurations are too numerous to list here. Only those references which are representative of the wide interest, numerous applications and of the variety of methods used in analyzing the problem are given here. The majority of research efforts thus far have been directed at the interfacial stability analysis of isothermal film with free interface (Benjamin 1957, Kapitza 1965; Yih 1963; Massot *et al.* 1966; Anshus & Goren 1966) or isothermal flow with adjacent gas phase (Lamb 1945; Long 1956) and have neglected the effects of heat transfer and interfacial phase change. On the other hand all investigations which were concerned with the effects of heat and mass transfer on the interfacial stability of liquid films neglected the effects of adjoining vapor flow (Bankoff 1971; Marshall & Lee 1973; Lin 1975; Unsal & Thomas 1978; Aleimikov 1979; Spindler 1982). Since excellent reviews on this topic have been given by Fulford (1964) and most recently by Solesio (1977), a detailed literature review is not attempted here. However, it is interesting to note that some inconsistencies seem to appear in the literature. As noted by Solesio (1977) and Spindler *et al.* (1978) most of the discrepancies stem from the fact that the kinematic and dynamic interfacial conditions used in some of the references were incorrect.

This paper has three purposes: (1) to present a two-fluid model formulation of a separated two-phase flow system which takes into account the simultaneous effects of vapor motion together with heat and mass transfer at the boundaries; (2) to develop from this formulation a general stability criterion so that conditions under which small disturbances will grow can be calculated by determining the neutral stability condition; (3) to apply the criterion to analyze the stability of the liquid film with interfacial shear and phase change.

TWO-FLUID MODEL FORMULATION

Field equations

The generalized separated two-phase flow system is illustrated in figure 1, where the two phases are distinguished by subscripts 1 and 2. Phases are flowing concurrently in a constant area duct with heat flux at the external boundaries and the phase change at the interface. The physical system is chosen in such a way that the resulting interfacial stability criterion can be used for the analysis of two-dimensional plane flow and axially symmetric annular two phase flow configurations.

Space-averaged field equations were originally proposed by Delhaye (1962) and Vernier & Delhaye (1968). The study of these equations and their application, within the frame of the two-fluid model, to a problem of liquid film hydrodynamics is given by Kocamustafaogullari (1971). According to the two-fluid modeling of two-phase flow systems, each phase is formulated in terms of two sets of field equations governing the kinematic, dynamic and energetic fields of each phase. A local formulation of the problem is given in appendix A. To simplify the problem, a detailed consideration of the relative magnitude of the various dimensionless parameters, a justification for neglecting various terms, and a discussion of appropriate scaling can also be found in the appendix. Integrating the simplified equations, [A24] and [A25], over the cross-sectional area of respective phases and using area-averaged variables, quasi-one-dimensional field equations governing each phase can be written as:

kinematic field equations

$$\frac{\partial}{\partial t}(1-\alpha)+\frac{\partial}{\partial x}[(1-\alpha)\langle u_1\rangle] = -(\dot{m}_{1i}/\rho_1)(\xi_i/A) \qquad [1]$$

$$\frac{\partial \alpha}{\partial t}+\frac{\partial}{\partial x}(\alpha\langle u_2\rangle) = -(\dot{m}_{2i}/\rho_2)(\xi_i/A) \qquad [2]$$

dynamic field equations

$$\rho_1\left(\frac{\partial\langle u_1\rangle}{\partial t}+\langle u_1\rangle\frac{\partial\langle u_1\rangle}{\partial x}\right) = -\frac{\partial\langle P_1\rangle}{\partial x}+\rho_1 g_x+\left(\frac{1}{1-\alpha}\right)$$

$$\times\left\{(P_{1i}-\langle P_1\rangle)\frac{\partial\alpha}{\partial x}-\dot{m}_{1i}(u_{1i}-\langle u_1\rangle)(\xi_i/A)+\tau_{1i}(\xi_i/A)\right.$$

$$\left.-\tau_{1e}(\xi_{1e}/A)-\frac{\partial}{\partial x}[(1-\alpha)\rho_1\mathrm{Cov}(u_1{}^2)]\right\} \qquad (3)$$

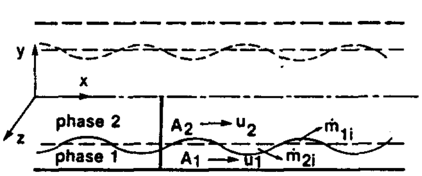

Figure 1. Typical separated two-phase flow (plane flow or annular flow).

$$\rho_2\left(\frac{\partial\langle u_2\rangle}{\partial t}+\langle u_2\rangle\frac{\partial\langle u_2\rangle}{\partial x}\right)=\frac{\partial\langle P_2\rangle}{\partial x}+\rho_2 g_x+\frac{1}{\alpha}\left\{(P_{2i}-\langle P_2\rangle)\frac{\partial\alpha}{\partial x}\right.$$
$$\left.-\dot{m}_{2i}(u_{2i}-\langle u_2\rangle)(\xi_i/A)-\tau_{2i}(\xi_i/A)-\tau_{2e}(\xi_{2e}/A)-\frac{\partial}{\partial x}\left[\alpha\rho_2\,\mathrm{Cov}(u_2^2)\right]\right\} \quad [4]$$

where subscripts i and e identify the internal and external boundaries whereas A, $\dot{m}$, P, u, α, ξ, ρ and τ are the total cross-sectional area of the flow channel, interfacial mass transfer per unit area per unit time, pressure, longitudinal velocity component, area based void fraction, perimeter, mass density and shear stress, respectively. Finally, $(\langle---\rangle)$ and Cov $(---)$ define the area-averaged value and covariance of a quantity. They are given as follows:

$$\langle F_k\rangle(x,t)\equiv\frac{1}{A_k}\int_{A_k}\int_{(x,t)}F_k(x,y,z,t)\,\mathrm{d}A;\quad k=1,2 \quad [5]$$

and

$$\mathrm{Cov}(F^2)\equiv\langle F^2\rangle-\langle F\rangle^2.$$

When [A26] is integrated over the cross-sectional area of each phase, the area-averaged pressures, $\langle P_1\rangle$ and $\langle P_2\rangle$, can be expressed in terms of the interfacial pressures, P_{1i} and P_{2i}. They are, respectively,

$$\langle P_1\rangle=P_{1i}-[(1-\alpha)\rho_1 g_y/2](A/\xi_i) \quad [6]$$

and

$$\langle P_2\rangle=P_{2i}-[\alpha\rho_2 g_y/2](A/\xi_i) \quad [7]$$

Interfacial balance equations

Since the macroscopic fields of one phase are not independent of the other phase, the interaction terms which couple the transport of mass and momentum of each phase across the interface appear in the field equations. Therefore, the two-fluid model requires knowledge of interfacial interactions.

A detailed statement of underlying simplifying assumptions and the derivations of interface balance equations are given in appendix A. These balance equations are reproduced here for ready reference:

mass balance

$$\dot{m}_{1i}+\dot{m}_{2i}=0 \quad [8]$$

momentum balance

$$P_{1i}-P_{2i}=\dot{m}_{1i}^2(\Delta\rho/\rho_1\rho_2)+(\sigma A/\xi_1)\frac{\partial^2\alpha}{\partial x^2} \quad [9]$$

$$\tau_{1i}-\tau_{2i}=0. \quad [10]$$

These balance equations must be supplemented by the kinematic no relative velocity condition at the interface.

$$u_{1i}=u_{2i}\equiv u_i. \quad [11]$$

The set of equations[1]–[11] presented here describe, within the frame of a two-fluid model, the separated two-phase flow of incompressible fluiods. However, subtracting[3] from [4], and using [5]–[11] in the resulting equation, the pressure terms can be completely eliminated. Thus,

$$\rho_2\left(\frac{\partial\langle u_2\rangle}{\partial t}+\langle u_2\rangle\frac{\partial\langle u_2\rangle}{\partial x}\right)-\rho_1\left(\frac{\partial\langle u_1\rangle}{\partial t}+\langle u_1\rangle\frac{\partial\langle u_1\rangle}{\partial x}\right)=\sigma(A/\xi_i)\frac{\partial^3\alpha}{\partial x^3}$$
$$+\left\{\Delta\rho g_y(A/\xi_i)-\left[\frac{\rho_2\,\mathrm{Cov}(u_2{}^2)}{\alpha}+\frac{\rho_1\,\mathrm{Cov}(u_1{}^2)}{1-\alpha}\right]\right\}\frac{\partial\alpha}{\partial x}-\Delta\rho g_x$$
$$+2(\Delta\rho/\rho_1\rho_2)\dot m_{1i}\frac{\partial\dot m_{1i}}{\partial x}+\left(\frac{u_i-\langle u_2\rangle}{\alpha}+\frac{u_i-\langle u_1\rangle}{1-\alpha}\right)\dot m_{1i}(\xi_i/A)$$
$$-\left(\frac{1}{\alpha}+\frac{1}{1-\alpha}\right)\tau_{1i}(\xi_i/A)-(\tau_{2e}/\alpha)(\xi_{2e}/A)+[\tau_{1e}/(1-\alpha)](\xi_{1e}/A). \qquad [12]$$

This equation replaces the dynamic field equations and eliminates $\langle P_1\rangle$, $\langle P_2\rangle$, P_{1i} and P_{2i} from the formulation.

Constitutive equations

Considering [1], [2] and [12] one notices that there are three basic dependent variables, α, $\langle u_1\rangle$ and $\langle u_2\rangle$ and seven supplementary variables, $\dot m_{1i}$ (or $\dot m_{2i}$), u_i, τ_{1i}, τ_{1e}, τ_{2e}, $\mathrm{Cov}(u_1{}^2)$ and $\mathrm{Cov}(u_2{}^2)$. In order to complete the formal formulation, the supplementary variables should be specified by the constitutive relations. They are completely dependent upon the flow regimes experienced by each phase. Therefore, depending upon the flow regime they can be related to the basic variables. In order to keep the analysis as general as possible one can express the functional relationship by

$$f=f(\alpha,\ \langle u_1\rangle,\ \langle u_2\rangle) \qquad [13]$$

where f stands for u_i, τ_{1i}, τ_{1e}, τ_{2e}, $\mathrm{Cov}(u_1{}^2)$ and $\mathrm{Cov}(u_2{}^2)$. Once the flow regimes are specified, then these supplementary variables can be specified through the constitutive relations in the form of [13]. On the other hand specification of $\dot m_{1i}$ or $\dot m_{2i}$ needs special attention.

By examining the kinematic field equations, it can be seen that the interfacial phase change represented by $\dot m_{1i}$ or $\dot m_{2i}$ acts as a sink; indeed, it plays the same role as the sink (or source) terms in the continuity equations of chemically reacting mixtures. Whereas in chemically reacting mixtures the sources (or sinks) are specified by appropriate constitutive equations of chemical kinetics, in two-phase flow they are specified by appropriate constitutive equations of phase change, i.e. of evaporation and condensation. It was shown by Zuber & Staub (1966) that the constitutive equations of evaporation (a) depend on the two-phase flow structure and (b) determine the thermodynamic non-equilibrium at the interface. Therefore, it will have a different form depending on whether the evaporation from the liquid film is affected by exposure to hot gases (as in rocket engines), or by heat transfer through the liquid film (as in boilers, evaporators or nuclear reactors). Considering this last application, the simplest expression for the constitutive equation of evaporation or condensation can be obtained by assuming that the vapor is saturated and that the thermal equilibrium exists at the interface. Furthermore, assuming a linear temperature distribution in the liquid film, the rate of phase change at the interface can be obtained directly from an energy balance. Thus,

$$\dot m_{1i}=-\dot m_{2i}=\epsilon\frac{k_1\Delta T}{h_{LG}(1-\alpha)}\frac{\xi_i}{A} \qquad [14]$$

where k_l is the thermal conductivity of liquid, h_{LG} is heat of vaporization, $\Delta T \equiv |T_w - T_s|$, and ϵ specifies the direction of phase change, $\epsilon = 1$ for evaporation and $\epsilon = -1$ for condensation.

An assumption related to the temperature distribution implies that the convection heat transfer within the liquid film is negligible. The conduction heat transfer essentially allows the phase change to take place. As shown by Sparrow & Gregg (1959) the linear temperature distribution is good for low values of the Kutateladze number, ($\mathrm{Ku} \equiv C_{pl}\Delta T/h_{LG}$). With increasing values of this parameter there are increasing deviations from linearity. As a consequence of neglecting the convection heat transfer, the analysis will be limited by $\mathrm{Ku} < 1$.

STABILITY ANALYSIS

In order to determine under what conditions waves appearing on the interface lead to instability, the behavior of very small perturbations will be examined on the perturbed flow equations. To obtain perturbed flow equations, the procedure is outlined as follows: (1) the basic flow variables, α, $\langle u_1\rangle$ and $\langle u_2\rangle$ written as:

$$F = \bar{F} + F' \qquad [15]$$

where $\bar{F}$ is the time-averaged mean value of any variable F and F' is its perturbation. (2) The supplementary variables are expressed by performing the Taylor Series expansion in[13] to obtain

$$f(\bar{\alpha} + \alpha', \langle \bar{u}_1\rangle + \langle u_1\rangle', (\bar{u}_2\rangle + \langle u_2\rangle') = \bar{f} + \frac{\partial \bar{f}}{\partial \bar{\alpha}}\alpha' + \frac{\partial \bar{f}}{\partial\langle \bar{u}_1\rangle}\langle u_1\rangle' + \frac{\partial \bar{f}}{\partial\langle \bar{u}_2\rangle}\langle u_2\rangle' + \mathrm{NT's} \qquad [16]$$

where NT's stand for the nonlinear perturbation terms. Similarly, constant wall temperature perturbations on the phase change are obtained from [14] as

$$\dot{m}_{1i}(\alpha + \alpha') + \frac{k_1\Delta T}{(1-\bar{\alpha})h_{LG}}\left(\frac{\xi_i}{A}\right)\left(1 + \frac{\alpha'}{1-\bar{\alpha}}\right) + \mathrm{NT's} \qquad [17]$$

(3) the perturbations given above are substituted in [1], [2] and [12]. Taking into account the mean flow equations and discarding all the nonlinear perturbation terms, the flow equations are linearized. Since the linearization is possible for long waves, (small amplitude in comparison with wave-length), the theory developed here will be applicable for long wave perturbations. (4) In a separated two-phase flow with an interfacial phase change, the mean flow variables, $\bar{F}$'s, change in the flow direction, x. However, it was demonstrated by Bankoff (1971) and Kocamustafaogullari (1971) that the effect of the phase change in the x-direction is very small for moderate values of wall heat flux. Therefore, one can consider the mean flow to be a quasi-fully developed flow in which the mean flow variables are not varying appreciable in the x-direction so that the multiplication of the perturbed quantities with the x-derivatives of the mean flow variables are assumed to be second order. To be consistent with the linearization these terms are neglected in comparison with the first-order effects.

Following the linearization procedure outlined above, the perturbed flow equations obtained from [1], [2] and [12], respectively, are

$$-\frac{\partial \alpha'}{\partial t} - \langle u_1\rangle\frac{\partial \alpha'}{\partial x} + (1-\bar{\alpha})\frac{\partial\langle u_1\rangle'}{\partial x} = -\epsilon\frac{k_1\Delta T}{\rho_1(1-\bar{\alpha})^2 h_{LG}}\left(\frac{\xi_i}{A}\right)\alpha' \qquad [18]$$

$$\frac{\partial \alpha'}{\partial t}+\langle u_2\rangle\frac{\partial \alpha'}{\partial x}+\bar{\alpha}\frac{\partial\langle u_2\rangle'}{\partial x}=\frac{\epsilon k_1 \Delta T}{\rho_1(1-\bar{\alpha})^2 h_{LG}}\left(\frac{\xi_i}{A}\right)\alpha' \qquad [19]$$

$$\rho_2\left(\frac{\partial\langle u_2\rangle'}{\partial t}+\langle\bar{u}_2\rangle\frac{\partial\langle u_2\rangle'}{\partial x}\right)-\rho_1\left(\frac{\partial\langle u_1\rangle'}{\partial t}+\langle\bar{u}_1\rangle\frac{\partial\langle u_1\rangle'}{\partial x}\right)$$

$$=a_1\frac{\partial^3\alpha'}{\partial x^3}+a_2\frac{\partial\alpha'}{\partial x}-a_3\frac{\partial\langle u_2\rangle'}{\partial x}+a_4\frac{\partial\langle u_1\rangle'}{\partial x}+a_5\alpha'-a_6\langle u_2\rangle'+a_7\langle u_1\rangle' \qquad [20]$$

where a's are defined in terms of the mean flow variables in appendix B.

The perturbed flow equations can be reduced into one. For this purpose, [20] is differentiated with respect to x, and [18] and [19] are used to express the derivatives of $\langle u_1\rangle'$ and $\langle u_2\rangle'$ in terms of those of α'. The resulting differential equation becomes

$$a_1\frac{\partial^4\alpha'}{\partial x^4}+\left[a_2-\left(\frac{a_3\langle\bar{u}_2\rangle}{\bar{\alpha}}+\frac{a_4\langle\bar{u}_1\rangle}{1-\bar{\alpha}}\right)+\left(\frac{\rho_2\langle\bar{u}_2\rangle^2}{\bar{\alpha}}+\frac{\rho_1\langle\bar{u}_1\rangle^2}{1-\bar{\alpha}}\right)\right]\frac{\partial^2\alpha'}{\partial x^2}$$

$$+\left[\left(\frac{a_3}{\bar{\alpha}}+\frac{a_4}{1-\bar{\alpha}}\right)+2\left(\frac{\rho_2\langle\bar{u}_2\rangle}{\bar{\alpha}}+\frac{\rho_1\langle\bar{u}_2\rangle}{1-\bar{\alpha}}\right)\right]\frac{\partial^2\alpha'}{\partial x\partial t}+\left(\frac{\rho_2}{\bar{\alpha}}+\frac{\rho_1}{1-\bar{\alpha}}\right)\frac{\partial^2\alpha'}{\partial t^2}$$

$$+\left\{\left[a_5+\left(\frac{a_6\langle\bar{u}_2\rangle}{\bar{\alpha}}\quad\frac{a_7\langle\bar{u}_1\rangle}{1-\bar{\alpha}}\right)-\left(\frac{a_3}{\rho_2\bar{\alpha}}+\frac{a_4}{\rho_1(1-\bar{\alpha})}\right)+\left(\frac{\langle\bar{u}_2\rangle}{\bar{\alpha}}+\frac{\langle\bar{u}_1\rangle}{1-\bar{\alpha}}\right)\right]\right.$$

$$\times\left(\frac{\epsilon k_1\Delta T}{(1-\bar{\alpha})^2 h_{LG}}\right)\left(\frac{\xi_i}{A}\right)^2\frac{\partial\alpha'}{\partial x}+\left[\left(\frac{a_6}{\bar{\alpha}}+\frac{a_7}{1-\bar{\alpha}}\right)-\left(\frac{1}{\bar{\alpha}}+\frac{1}{1-\bar{\alpha}}\right)\right.$$

$$\left.\left.\times\left(\frac{\epsilon k_1\Delta T}{(1-\bar{\alpha})^2 h_{LG}}\right)\left(\frac{\xi_i}{A}\right)^2\right]\right\}\frac{\partial\alpha'}{\partial t}$$

$$-\left(\frac{a_6}{\rho_2\bar{\alpha}}+\frac{a_7}{\rho_1(1-\bar{\alpha})}\right)\left(\frac{\epsilon k_1\Delta T}{(1-\bar{\alpha})^2 h_{LG}}\right)\left(\frac{\xi_i}{A}\right)\alpha'=0 \qquad [21]$$

This differential equation is the characteristic equation from which the stability of the system under consideration can be determined. Now, the task of the stability theory consists in determining whether the disturbance amplifies or decays for a given time-averaged mean flow.

Stability criterion

The mean flow with a void fraction of $\bar{\alpha}$ is assumed to be influenced by a disturbance which is composed of a number of discrete partial fluctuations, each of which consists of a wave propagation in the mean flow direction. Therefore, analyzing the disturbance into normal modes, we seek a solution whose dependency on x and t is given by

$$\alpha'=\alpha_i\exp[ik(x-ct)] \qquad [22]$$

where c is the complex wave celerity, k is the wave number which is related to the wave length by $k=2\pi/\lambda$, and α_i is the perturbation void fraction amplitude.

The mean flow depends on the abscissa x. As a consequence, from a mathematical point of view, a perturbation amplitude, α_i, must be a function of x, the multiple scale method. However, to be consistent with the quasi-fully developed flow approximation introduced in the previous section, the x-dependency of the characteristics of one perturbation is no longer considered, the local approximation method. In this method the

x-dependent flow is replaced by its local value at a fixed abscissa, (Spindler 1981). However, the method developed here permits the calculation of the characteristic value of a perturbation initiated at a given x.

Introducing [22] in [21], the dispersion relation is obtained and given by

$$[kc - (b_1k + b_2i)]^2 = (b_1k)^2 - b_2^2 + b_3 + (2b_1b_2 + b_4)ki \qquad [23]$$

where for the purpose of convenience parameters b's are introduced. Defining equations for b's are given in appendix B.

Letting $c \equiv c_r + ic_i$, [23] can be rearranged in a more convenient form as

$$(C_r + iC_i)^2k^2 = U + iV \qquad [24]$$

where

$$\begin{aligned} C_r &\equiv c_r - b_1 \\ C_i &\equiv c_i - b_2/k \\ U &\equiv (b_1k)^2 - b_2^2 + b_3 \\ V &\equiv (2b_1b_2 + b_4)k \end{aligned} \qquad [25]$$

It can be shown that U and V are conjugate harmonic functions, (Churchill 1960). Their contour curves, $U = $ const. and $V = $ const., are those shown in figure 2. It is clear from the figure that for each value of the harmonic functions U and V, there exists a pair of solutions for C_r and C_i. Although these solutions are equal in numerical value they are different in sign. This situation is not surprising because we are analyzing the dynamic waves which propagate in downstream as well as in upstream directions. A positive value of C_r, therefore, corresponds to the wave train propagating downstream with respect to b_1, whereas a negative value of C_r corresponds to the wave propagating upstream. As a

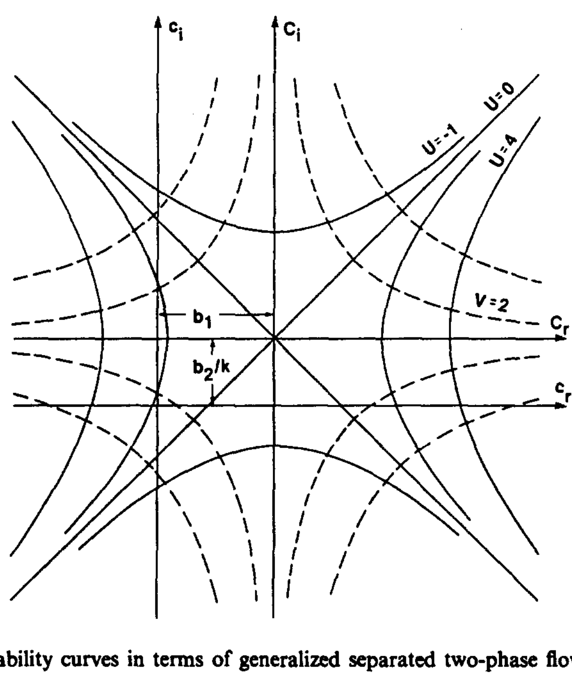

Figure 2. Stability curves in terms of generalized separated two-phase flow parameters.

result it can be concluded that the right hand side of figure 2 corresponds to downstream moving waves whereas the left side corresponds to upsteam moving waves.

Solving [24], the speed of propagation of waves and the growth factor are, respectively, obtained as follows:

$$C_r \equiv c_r - b_1 = \pm\frac{1}{k}\left[\frac{(U^2+V^2)^{\frac{1}{2}}+U}{2}\right]^{\frac{1}{2}} \qquad [26]$$

and

$$kc_i \equiv kc_i - b_2 = \pm\left[\frac{(U^2+V^2)^{\frac{1}{2}}-U}{2}\right]^{\frac{1}{2}}. \qquad [27]$$

Infinitesimal perturbations on the mean value of void fraction will grow or decay in amplitude depending on whether $c_i > 0$ or $c_i < 0$. The neutral stability condition ($c_i = 0$) separates the stable from the unstable region. It is evident from figure 2 and [27] that the positive values of kC_i are more dangerous than the negative values. Therefore, the discussion can be confined for the case where $kC_i > 0$. In this case, if $b_2 < 0$, kc_i is always positive indicating that the flow is unstable. Therefore, the necessary (but not the sufficient) condition for the stability of the system should be given by

$$b_2 < 0. \qquad [28]$$

The over-all stability criterion follows immediately from [27] as

$$kc_i = b_2 + \left[\frac{(U^2+V^2)^{1/2}-U}{2}\right]^{1/2} \leq 0 \qquad [29]$$

which in view of [23] can be expressed as

$$(b_4/2b_2)^2k^2 + 2(b_4/2b_2)b_1k^2 - b_3 \leq 0 \qquad [30]$$

where the equality sign stands for the neutral stability condition. Equation [28], together with [29] or [30], yields the stability criteria of interfacial waves for a separated two-phase flow system with phase change and interfacial shear. The criteria were established in terms of the generalized parameters of the separated two-phase flow system. Therefore, one can use the criteria to analyze the Kelvin–Helmholtz instability, the Rayleigh–Taylor instability or the stability of a free film flow provided the parameters such as a's and b's defined in appendix B are evaluated properly. We shall apply what follows to the criteria to analyze the stability of laminar film flow with interfacial shear and phase change.

APPLICATION

Stability of laminar liquid film flow with interfacial shear and phase change

Mean flow parameters. The basic flow, the stability of which is to be investigated, is the laminar flow of a thin film of a Newtonian liquid with constant properties under the action of gravity and adjacent vapor flow. Here the liquid film is bounded on the one side by the solid plate and on the other by an interface subject to a constant interfacial shear exerted by its saturated vapor, figure 3. The plate is maintained at constant temperature, T_w. Furthermore, it is assumed that the liquid film thickness is much smaller than the vapor layer thickness so that one can approximate the concentration ration by

$$\frac{1-\bar{\alpha}}{\bar{\alpha}} \to 0. \qquad [31]$$

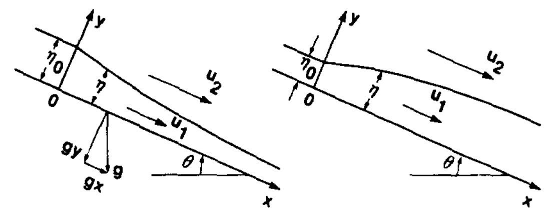

Figure 3. Plane liquid film flow with phase change (Spindler 1982).

Under the assumptions listed above, the base flow momentum and continuity equations, i.e. the zeroth order terms of $(e_1 \,\mathrm{Re}_1)$ in [A16] and [A17], are used to obtain the base flow velocity and film thickness profiles. They are given, respectively, by Rohsenow *et al.* (1956), as follows:

$$\bar{u}_1 = \left(\frac{\Delta\rho g_x}{\mu_1}\right)\left(\bar{\eta}y - \frac{y^2}{2}\right) + \frac{\bar{\tau}_{1i}y}{\mu_1} \qquad [32]$$

and

$$\bar{\eta}^4 - \bar{\eta}_0^4 + \frac{4}{3}\left(\frac{\bar{\tau}_{1i}}{\Delta\rho g_x}\right)(\bar{\eta}^3 - \eta_0^3) = -\epsilon\left(\frac{4\mu_1 k_1 \Delta T}{\rho_1 \Delta\rho g_x h_{LG}}\right)x \qquad [33]$$

where η_0 is the mean film thickness at $x = 0$. For evaporation $\eta_0 \neq 0$. For condensation without pre-existing liquid film, $\eta_0 = 0$.

Averaging u_1 over the local film thickness η, it is readily shown that

$$\langle \bar{u}_1 \rangle = \left(\frac{\Delta\rho g_x}{3\mu_1}\right)\bar{\eta}^2 + \left(\frac{\bar{\tau}_{1i}}{2\mu_1}\right)\bar{\eta} \qquad [34]$$

Furthermore, in view of [32] and [34], the mean flow supplementary variables u_{1i}, τ_{1e} and $\mathrm{Cov}(u_1{}^2)$ can be expressed in terms of the basic flow variables η and $\langle u_1 \rangle$. They are as follows:

$$u_{1i} = \frac{3}{2}\langle \bar{u}_1 \rangle + \frac{1}{4}\frac{\bar{\tau}_{1i}\bar{\eta}}{\mu_1} \qquad [35]$$

$$\bar{\tau}_{1e} = 3\frac{\mu_1\langle \bar{u}_1 \rangle}{\bar{\eta}} + \frac{5}{3}\bar{\tau}_{1i} \qquad [36]$$

$$\mathrm{Cov}(u_1{}^2) = \frac{1}{3}\langle \bar{u}_1 \rangle^2 + \frac{1}{30}\left(\frac{\bar{\tau}_{1i}\bar{\eta}}{\mu_1}\right)^2 + \frac{1}{60}\left(\frac{g_x \Delta\rho\bar{\eta}^2}{\mu_1}\right)\left(\frac{\bar{\tau}_{1i}\bar{\eta}}{\mu_1}\right) \qquad [37]$$

Stability criteria. In view of [31, 33–37], the parameters a's and b's defined by [B1]–[B11] are calculated in terms of the basic flow parameters. Using these parameters, the stability conditions are calculated from [28] and [30] whereas the speed of propagation and the growth factor are determined from [26] and [27], respectively. The resulting equations are expressed in terms of appropriate dimensionless parameters. The results are as follows:

the necessary condition

$$\epsilon \,\mathrm{Ku}/6\,\mathrm{Pr} < 1 \qquad [38]$$

the dimensionless overall stability condition

$$\mathrm{Ka}\,\eta^{*3}k^{*4} - \left\{ g^*\eta^{*3} + 2\rho^*\left(\frac{\mathrm{Ku}}{\mathrm{Pr}}\right)^2 + \frac{6}{5}\mathrm{Re}^2\left(1 - \frac{1}{6}\frac{\tau_i^*\eta^{*2}}{\mathrm{Re}}\right) \right.$$

$$\left. + 9\,\mathrm{Re}^2\left[\frac{1+\dfrac{\epsilon}{90}\left(\dfrac{\mathrm{Ku}}{\mathrm{Pr}}\right)}{1-\dfrac{\epsilon}{6}\left(\dfrac{\mathrm{Ku}}{\mathrm{Pr}}\right)} - \frac{1}{6}\frac{\tau_i^*\eta^{*2}}{\mathrm{Re}}\right]^2 - \frac{36}{5}\mathrm{Re}^2\left[\frac{1+\dfrac{\epsilon}{90}\left(\dfrac{\mathrm{Ku}}{\mathrm{Pr}}\right)}{1-\dfrac{\epsilon}{6}\left(\dfrac{\mathrm{Ku}}{\mathrm{Pr}}\right)} - \frac{\tau_i^*\eta^{*2}}{6\,\mathrm{Re}}\right]\right\}k^{*2}$$

$$-\frac{3\epsilon}{\eta^{*2}}\left(1+\frac{\epsilon\,\mathrm{Ku}}{6\,\mathrm{Pr}}\right)\left(\frac{\mathrm{Ku}}{\mathrm{Pr}}\right) \geq 0 \qquad [39]$$

the dimensionless wave velocity

$$c_r^* = \frac{6}{5} + \left[\frac{(U^{*2}+V^{*2})}{2}\right]^{1/2} \qquad [40]$$

the dimensionless temporal growth factor

$$k^*c_i^* = -\frac{3}{2}\left(\frac{1-\dfrac{\epsilon\,\mathrm{Ku}}{6\,\mathrm{Pr}}}{\eta^*\,\mathrm{Re}}\right) + \left[\frac{(U^{*2}+V^{*2})^{1/2} - U^*}{2}\right]^{1/2} \qquad [41]$$

where U^* and V^* are defined as

$$U^* = \frac{1}{\mathrm{Re}^2}\left\{\mathrm{Ka}\eta^{*3}k^{*4} - \left[g^*\eta^{*3} - \frac{6}{25}\mathrm{Re}^2 - \frac{1}{15}\tau_i\eta^{*5}\left(1+\frac{3\tau_i^*}{2\eta^*}\right)\right.\right.$$

$$\left.\left. + 2\rho^*\left(\frac{\mathrm{Ku}}{\mathrm{Pr}}\right)^2\right]k^{*2} - 3\epsilon\left(1+\frac{1}{6}\frac{\mathrm{Ku}}{\mathrm{Pr}}\right)\left(\frac{\mathrm{Ku}}{\mathrm{Pr}}\right)\frac{1}{\eta^{*2}} - \frac{9}{4}\left(1-\frac{\epsilon\,\mathrm{Ku}}{6\,\mathrm{Pr}}\right)^2\frac{1}{\eta^{*2}}\right\} \qquad [42]$$

$$V^* = \frac{1}{\mathrm{Re}^2}\left[\frac{27}{5}\left(1+\frac{7\epsilon\,\mathrm{Ku}}{54\,\mathrm{Pr}}\right)\frac{\mathrm{Re}}{\eta^*} - \frac{3}{2}\left(1-\frac{\epsilon\,\mathrm{Ku}}{6\,\mathrm{Pr}}\right)\tau_i^*\eta^*\right]k^*. \qquad [43]$$

In these equations, dimensionless quantities, i.e., the Kapitza number Ka, Kutateladze number Ku, Prandtl number Pr, Reynolds number Re, film thickness parameter η^*, wave number k^*, interfacial shear τ_i^*, gravity parameter g^*, density ratio parameter ρ^*, dimensionless wave velocity c_r^* and dimensionless temporal growth factor c_i^*, are all defined as follows:

$$\begin{aligned}
&\mathrm{Ka} \equiv (\sigma^3\rho_1{}^2/\mu_1{}^4\Delta\rho g_x)^{1/3}; && \tau_i^* \equiv (\tau_i/\Delta\rho\eta g_x)\eta^* \\
&\mathrm{Ku} \equiv C_{p1}\Delta T/h_{LG}; && g^* \equiv g_y/g_x = -\cot g\theta \\
&\mathrm{Pr} \equiv C_{p1}\mu_1/k_1; && \rho^* \equiv (\rho_1-\rho_2)/\rho_2 \\
&\mathrm{Re} \equiv \bar{\rho}_1\eta\langle\bar{u}_1\rangle/\mu_1; && c_r^* \equiv c_r/\langle\bar{u}_1\rangle \\
&\eta^* \equiv \eta(\rho_1\Delta\rho g_x/\mu_1{}^2)^{1/3}; && c_i^* \equiv c_i/\langle\bar{u}_1\rangle \\
&k^* \equiv k(\mu_1{}^2/\rho_1\Delta\rho g_x)^{1/3}
\end{aligned} \qquad [44]$$

When expressed in terms of dimensionless parameters, the base flow solutions, [34] and

[35], respectively, become

$$\mathrm{Re} = \frac{1}{3}\eta^{*3} + \frac{1}{2}\tau_i^*\eta^{*2} \qquad [45]$$

$$4\epsilon x^* = \eta_0^{*4} - \eta^{*4} + \frac{4}{3}\tau_i^*(\eta_0^{*3} - \eta^{*3}) \qquad [46]$$

where x^* is the dimensionless distance, $(x^* \equiv x(\mathrm{Ku}/\mathrm{Pr})(\rho_1 \Delta\rho g_x/\mu_1{}^2)^{1/3}$.

In view of [45] only two of the dimensionless quantities (Re, η^*, τ_i^*) can be chosen independently. Although the Re number is a proper one for the liquid film flow free of interfacial shear ($\tau_i^* = 0$) analysis, η^* is more appropriate for the film flow with interfacial shear because it represents the sole effect of film thickness. Equation [46] serves to find the location in terms of film thickness and interfacial shear.

RESULTS AND DISCUSSION

Necessary condition. The necessary condition expressed by [38] reveals the region where the stability analysis developed here holds. In the case of condensation ($\epsilon = -1$), it is identically satisfied indicating that there exists an absolutely stable film flow region. On the other hand, for evaporation ($\epsilon = 1$), it seems that the condition is not satisfied identically. However, numerical evaluation of Kutateladze number and of (Ku/Pr) for common fluids (Pr > 1), and even for liquid metals ($Cp_1/h_{LG} \ll 1$, and Pr < 1), indicates that for practical purposes the necessary condition is always satisfied. Consequently, the discussion will be limited to fluids for which (Ku/Pr) < 1.

The over-all stability condition. With the limitation imposed above and using [45], the overall stability condition is simplified with negligible loss of accuracy to

$$\mathrm{Ka}\,\eta^{*3}k^{*4} - \left[g^*\eta^{*3} + 2\rho^*\left(\frac{\mathrm{Ku}}{\mathrm{Pr}}\right)^2 + \frac{1}{3}\eta^{*6}\left(1 + \frac{\tau_i^*}{\eta^*}\right)\right]k^{*2} - \left(\frac{3}{\eta^{*3}}\right)\left(\frac{\mathrm{Ku}}{\mathrm{Pr}}\right) \geq 0 \qquad [47]$$

The effects of different parameters on the stability is evident in [47]. Before going into numerical evaluation of this criterion, several qualitative conclusions can be drawn from it by analyzing the sign of different terms. (1) It is clear that the surface tension parameter Ka has a stabilizing effect. However, the presence of k^{*4} in the same term indicates that the surface tension effect is diminished for sufficiently small values of the wave number, i.e., for very long waves, which has been well established in the literature, (Benjamin 1957; Yih 1963). (2) Now considering the gravity parameter g^*, it is evident that if $\theta < 90°$, then $g^* = -\cot g\,\theta < 0$, (see figure 3). Therefore, the gravity has a stabilizing effect, whereas it has a destablizing effect for $\theta > 90°$. (The Rayleigh–Taylor Instability as demonstrated by Chandrasekhar 1961.) (3) The film thickness parameter η^* and the interfacial shear parameter τ_i^* both have destabilizing effects. (4) In the case of evaporation ($\epsilon = 1$) it is evident that the phase change parameter Ku has a destablizing effect. In the case of condensation ($\epsilon = -1$), it has both stabilizing and destabilizing effects. The term containing $(\mathrm{Ku}/\mathrm{Pr})^2$ has a destabilizing effect because it has a negative sign whereas the last term has a stabilizing effect because its sign becomes positive for $\epsilon = -1$. This tentative conclusion agrees well with the result of Unsal & Thomas (1978).

The relative order of these opposite effects of condensation can be assessed by comparing these two terms. Hence, if

$$\eta^*k^* \equiv \eta k < [(3/2\rho^*)(\mathrm{Pr}/\mathrm{Ku})]^{1/2} \qquad [48]$$

then the stabilizing effect overcomes the destabilizing one. This is the case for very long waves. For example, for flow of saturated water at 100°C ($\rho_1 = 9.6 \times 10^2$ kg m^{-3}; $\rho_2 = 0.6$ kg m^{-3}; $h_{LG} = 2.3 \times 10^6$ Jkg^{-1}; $k_1 = 6.8 \times 10^{-1}$ Wm^{-1} K^{-1}; $\mu_1 = 2.8 \times 10^{-4}$ kg m^{-1} s^{-1}), with $\Delta T = 5.5$ K (Ku $= 10^{-2}$) [48] yields $\eta k < 0.4$ which can be satisfied for only sufficiently small wave numbers, i.e. for long waves. Since the linearized stability theory holds for the long wave disturbances, the destabilizing effect can be discarded in the region where the analysis applies. It is concluded, therefore, that the condensation phase change has a dual role in the stability mechanism with the stabilizing effect dominating over the destabilizing effect for sufficiently long waves.

Neutral stability curves. The quantitative effects of the various parameters on the stability are studied by obtaining the neutral stability curves. By setting [47] to zero and solving the resulting equation for the wave number yields

$$k^* = \left(\frac{1}{2Ka}\right)\left\{g^* + 2\left(\frac{\rho^*}{\eta^{*3}}\right)\left(\frac{\mathrm{Ku}}{\mathrm{Pr}}\right)^2 + \frac{1}{3}\eta^{*3}\left(1 + \frac{\tau_i}{\eta^*}\right)\right.$$
$$\left. \pm\left[\left(g^* + 2\left(\frac{\rho^*}{\eta^{*3}}\right)\left(\frac{\mathrm{Ku}}{\mathrm{Pr}}\right)^2 + \frac{1}{3}\eta^{*3}\left(1 + \frac{\tau_i^*}{\eta^*}\right)\right)^2 + 12\epsilon\left(\frac{\mathrm{Ka}}{\eta^{*5}}\right)\left(\frac{\mathrm{Ku}}{\mathrm{Pr}}\right)\right]^{1/2}\right\}^{1/2} \qquad [49]$$

In view of the approximate limit of laminar liquid film theory (Rohsenow *et al.* 1956), the influence of the following parameters on the stability has been studied on [49].

(1) Phase change parameter, $5 \times 10^{-3} \leq \mathrm{Ku} \leq 5 \times 10^{-2}$.
(2) Film thickness parameter, $1 < \eta^* \leq 10$.
(3) Gravity parameter, $-5.67 \leq g^* \leq 0 (10° < \theta < 90°)$.
(4) Interfacial shear parameter, $0 \leq \tau_i^* \leq 20$.

For a vertical ($g^* = 0$) water film flow free of interfacial shear ($\tau_i^* = 0$), the neutral stability curves predicted by [49] are presented in figures 4 and 5 for selected values of interfacial phase change parameter, Ku. Re $= \frac{1}{3}\eta^{*3}$ for $\tau_i^* = 0$. Therefore, η^{*3}, instead of η^*, itself is chosen as an abscissa in these figures. This facilitates the comparison of the analytical results of this investigation with those of other authors, where Reynolds number has been used to represent the influence of the film thickness. An inspection of these figures clearly shows the effect of the interfacial phase change parameter. It promotes instability in evaporating liquid films (figure 4), and stabilizes in condensate film flow (figure 5). Either in evaporation or condensation, the phase change influence increases as the Kutateladze number becomes larger, and its effect becomes more significant at sufficiently low values of η^*. As η^* increases the effect of the interfacial phase change diminishes, and the neutral stability curves appear to converge to the curve with no interfacial phase change (Ku = 0). However, complete convergence cannot be achieved because of the thermophysical properties of the fluid. In the present analysis the properties are evaluated at the reference temperature $(T_w + T_s)/2$ except for surface tension and latent heat which are evaluated at the interface temperature (saturation temperature). Therefore, depending on the value for the Kutateladze number, the liquid properties; hence, the Kapitza number changes. For example, in the case of evaporating liquid films ($T_w > T_s$), increasing Ku number increases the reference temperature and hence increases the Ka number, which reduces the destabilizing effect of the Kutateladze number. This is clearly seen in figure 4 for sufficiently high values of the wave number. On the other hand behavior of the Kapitza number is reversed in the case of condensation. This explains why the neutral stability curves cannot approach completely to the isothermal curve. Finally, it is to be noticed in these figures

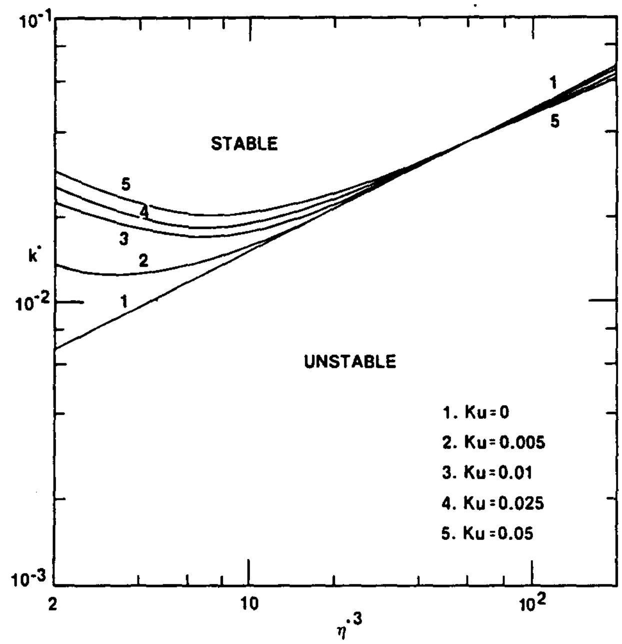

Figure 4. Neutral stability curves, evaporating water film flow at 373 K; $g^* = 0$, $\tau_i^* = 0$.

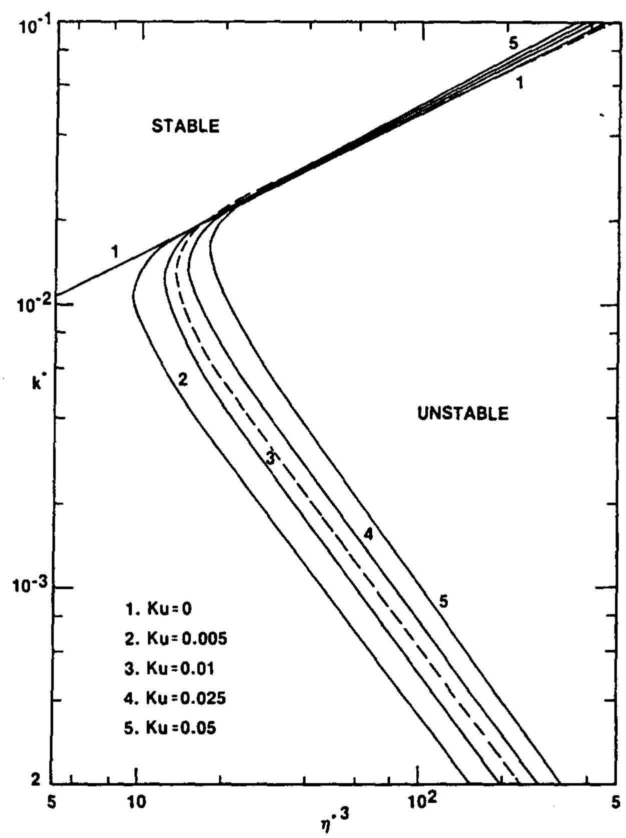

Figure 5. Neutral stability curves, condensing water film flow at 373 K; $g^* = 0$, $\tau_i^* = 0$. ——, present study; ---, Spindler (1982) for Ku = 0.01.

that the stability is greatly affected by the film thickness parameter, η^*. The thicker the film the more unstable it becomes. As the distance x increases, a thickness decrease (evaporation) has a stabilizing effect whereas a thickness increase (condensation) has a destabilizing effect. A unique relation between the distance and the film thickness parameter is given by [46].

As a check on the accuracy of this method, also shown in figure 5 is the neutral stability curve predicted by Spindler (1982), who used local equations to analyze the influence of interfacial phase change on the stability of liquid film flow free of interfacial shear. The agreement between Spindler's more accurate results and the present calculations is thought to be quite good.

In the case of condensation, figure 5 shows that there exists a critical film thickness parameter η_c^* below which the flow is completely stable indicating that the condensation induces an absolute stability. Its value can be determined from [49] by setting the discriminant to zero. With $\epsilon = -1$, it yields.

$$\eta_c^{*5/2}\left[g^* + 2\left(\frac{\rho^*}{\eta_c^*}\right)\left(\frac{\mathrm{Ku}}{\mathrm{Pr}}\right)^2 + \frac{1}{3}\eta_c^{*3}\left(1+\frac{\tau_i^*}{\eta_c^*}\right)\right] = 2\left[3\left(\frac{\mathrm{Ku}}{\mathrm{Pr}}\right)\mathrm{Ka}\right]^{1/2}. \qquad [50]$$

The critical wave number k_c^* becomes

$$k_c^* = \left(\frac{1}{2\mathrm{Ka}}\right)^{1/2}\left[g^* + 2\left(\frac{\rho^*}{\eta_c^{*3}}\right)\left(\frac{\mathrm{Ku}}{\mathrm{Pr}}\right)^2 + \frac{1}{3}\eta_c^*\left(1+\frac{\tau_i^*}{\eta_c^*}\right)\right]^{1/2}. \qquad [51]$$

The comparison of [50] with those of other authors, Unsal and Thomas and of Spindler, can be made only for a vertical ($g^* = 0$) condensing film flow with no interfacial shear ($\tau_i^* = 0$). In this case [50] reduces to

$$\eta_c^* = 1.54(\mathrm{Ka\,Ku/Pr})^{1/11}. \qquad [52]$$

When expressed in terms of the nomenclature used in the present study, the critical film thickness calculated by these references can be expressed exactly the same as the form of [52], apart from replacing the coefficient 1.54 in [52] by 1.48 in the former reference and by 1.68 in the latter one. This is a deviation of 4% from Unsal and Thomas' solution, and of 5.5% from Spindler's solution. The differences in the coefficient can be explained by the different set of field equations used: Local equations (Unsal & Thomas 1978; Spindler 1982) or equations averaged over the cross sectional area (present study). The real advantage of the present analysis is that it is simple and general and, thus, can be used in the investigation of the stability of other liquid film flows in the presence of vapor flow.

For a vertical ($g^* = 0$) flow free of interfacial shear ($\tau_i^* = 0$) condensation initiating at the leading edge of the plate ($\eta_0^* = 0$), the combination of [52] and [46] yields the critical distance x_c^*. Thus,

$$x_c^* = 1.41(\mathrm{Ka\,Ku/Pr})^{4/11} \qquad [53]$$

This equation states that the flow is always stable for perturbations born at an abscissa lower than the critical distance x_c^*, for any wave number.

Simultaneous effects of the phase change parameter Ku and the gravity parameter g^* are illustrated in figures 6 and 7 for a non-vertical flow free of interfacial shear. As can be seen from these figures the gravity parameter has a clear stabilizing effect when $0 < \theta < 90°$. However, within the limitations imposed by the assumption made in the

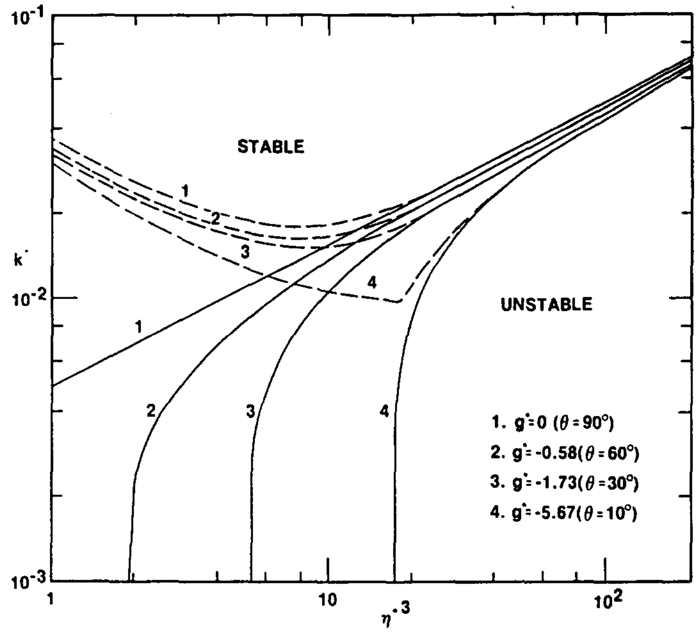

Figure 6. Neutral stability curves, evaporating water film flow at 373 K; $\tau_i^* = 0$. ——, Ku = 0; - - -, Ku = −0.01.

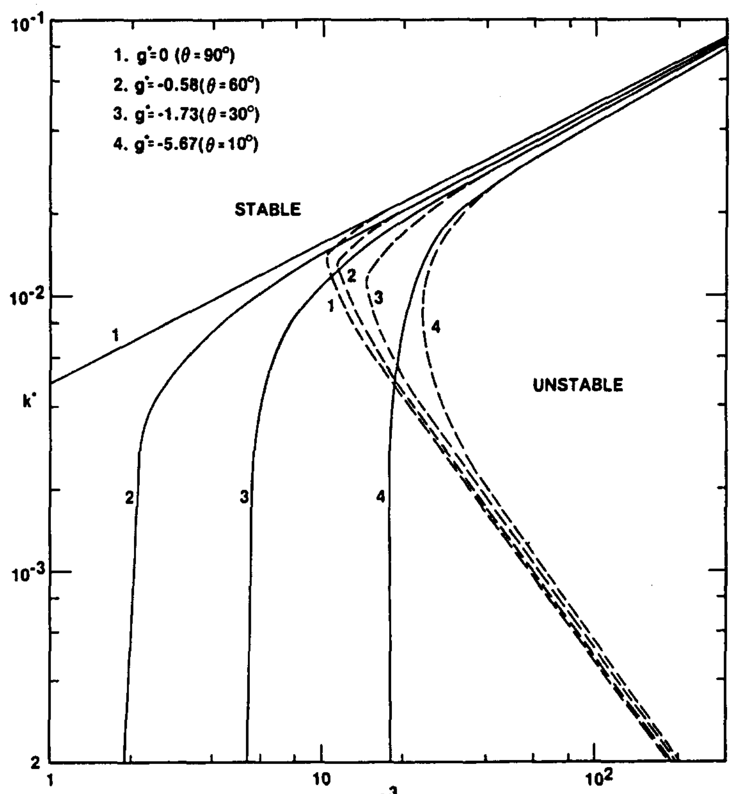

Figure 7. Neutral stability curves, condensing water film flow at 373 K; $\tau_i^* = 0$. ——, Ku = 0; - - -, Ku = 0.01.

analysis the gravity cannot induce complete stability in an evaporating liquid film flow, although, it may stabilize waves whose wave numbers are greater than a certain value (figure 6). On the other hand in the case of condensation simultaneous stabilizing influence of both gravity and phase change is demonstrated by the fact that the stable region increases as g^* decreases, and in particular by the fact that the value of the critical thickness parameter η_c^* increases as g^* decreases (figure 7). For non-vertical flow, η_c^* can be calculated from [50]. When $g^* \neq 0$, a quick check of this equation can be made with that of Yih (1963), who sought solutions to the Orr–Sommerfeld equation as a power of the wave number k^*. For sufficiently small values of the wave number, Yih's solution gives $\eta_c^* = 1.36\,(\mathrm{cotg}\,\theta)^{1/3}$. It has to be compared with the following relation established from [50] for Ku = 0 and $\tau_i^* = 0$:

$$\eta_c^* = -1.44\,g^{*1/3} = 1.44\,(\mathrm{cotg}\,\theta)^{1/3} \qquad [54]$$

It is to be noted that this result compares favorably with those obtained by Yih, thereby again giving support to the simple analysis with integral equations.

The destabilizing effect of vapor flow on the stability of a vertical water film flow at Ku = 0.01 is demonstrated in figure 8. This figure indicates that the increase of interfacial shear exerted by the vapor flow can considerably alter the neutral stability curves and critical film thickness η_c^*. For example, for a vertical water flow at 373 K (Ku = 0.01), [50] yields $\eta_c^* = 2.28$ for $\tau_i^* = 0$ and $\eta_c^* = 0.84$ for $\tau_i^* = 10$. Therefore, the destabilizing effect of condensation at low thickness parameter is almost completely overcome by the destabilizing effect of the interfacial shear.

Temporal growth factor. The temporal growth factor $k^*c_i^*$ is compared in figures 9 and 10 for a wide range of the basic parameters Ku and η^*. As indicated in figure 9, in the

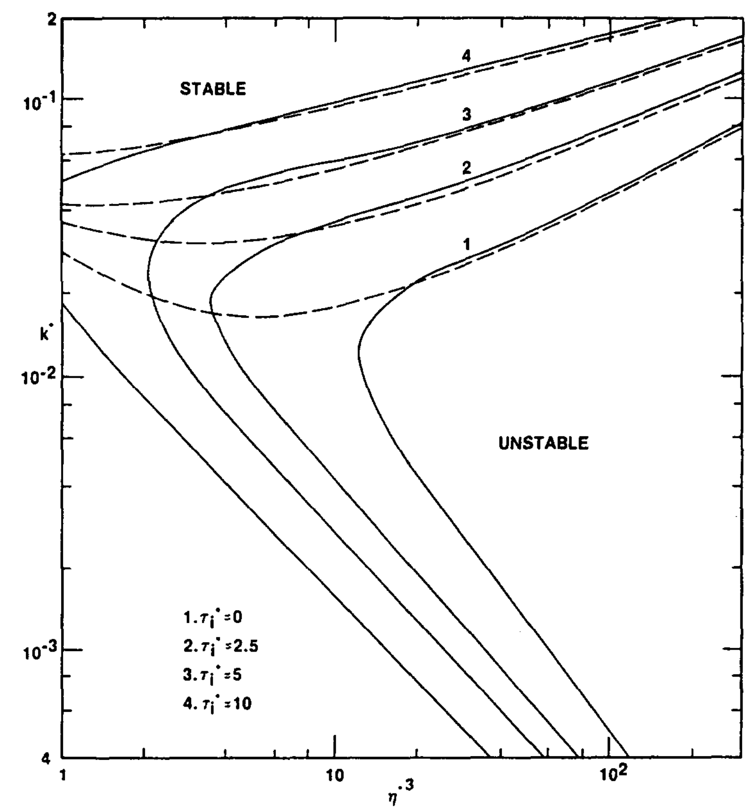

Figure 8. Neutral stability curves, water film flow at 373 K; Ku = 0.01, $g^* = 0$. ——, condensation: – – –, evaporation.

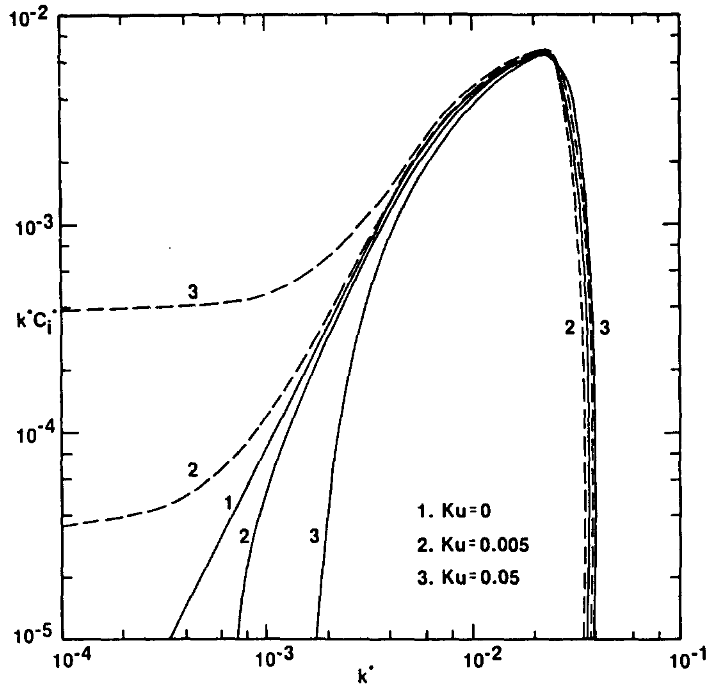

Figure 9. Growth factor vs wave number, vertical film flow of water at 373 K, $g^* = 0$, $\tau_i^* = 0$, $\eta^* = 4$. ——, condensation; - - -, evaporation.

case of evaporation the growth factor increases with the phase change parameter (destabilizing effect), whereas in the case of condensation it decreases with the phase change parameter (stabilizing effect). The phase change has almost no influence on the maximum growth factor and on the disturbances whose wave number is greater than the one corresponding to the maximum growth factor. As can be seen from figure 10 the maximum growth factor has a maximum when regarded as a function of the film thickness

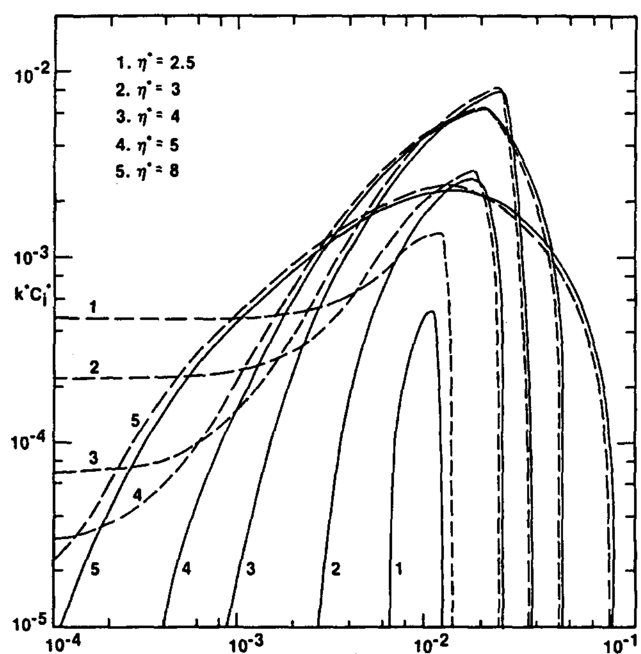

Figure 10. Growth factor vs wave number, vertical film flow of water at 373 K, $g^* - 0$, $\tau_i^* = 0$, Ku = 0.01. ——, condensation; - - -, evaporation.

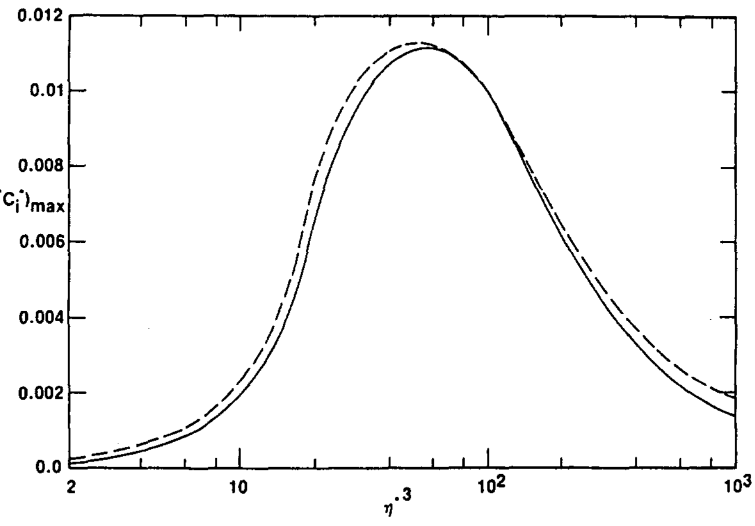

Figure 11. Maximum growth factor as calculated from the present analysis compared with the growth factor as predicted by Anshus & Goren, isothermal flow of water at 293 K (Ku = 0, $g^* = 0$, $\tau_i^* = 0$). ——, present study; - - -, Anshus & Goren (1966).

parameter. This means that the rate of growth of small disturbances increases with an increasing film thickness parameter for low film thicknesses (less than about 4 for water at 373 K) but decreases with increasing film thickness at high film thicknesses. The same conclusion was reached by Anshus & Goren (1966) who obtained approximate solutions to the Orr–Sommerfeld equation for isothermal film flow free of interfacial shear. To check the accuracy of this method the maximum growth factor obtained from [41] is compared with the maximum growth factor curve supplied by Anshus & Goren. The results as a function of the film thickness parameter are shown in figure 11 for a vertical flow of water at 20°C ($g^* = 0$, $\tau_i^* = 0$, Ku = 0). The two calculations show very good agreement.

Figure 12 demonstrates the influence of the interfacial shear parameter. It is interesting to note that the growth factor for this example goes through a maximum at $k^* \simeq 2 \times 10^{-2}$. At present we do not have a physical interpretation of this prediction or the data to check

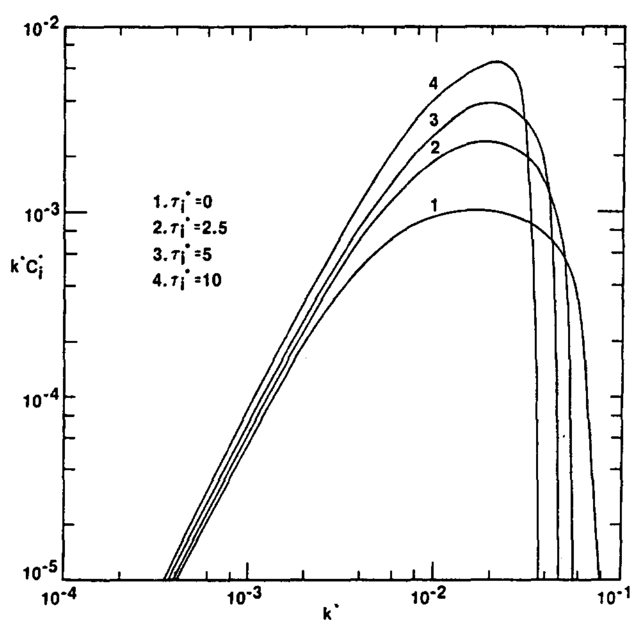

Figure 12. Growth factor vs wave number, vertical film flow of water at 373 K. $g^* = 0$, Ku = 0, $\eta^* = 4$.

it. Furthermore, when interpreted together with figure 8 it is to be noticed that as the interfacial shear parameter increases waves initiate in a wider band of wave number (destabilizing effect). However, the growth factor of the most unstable wave decreases with interfacial shear parameter (smoothing effect of the interfacial shear).

Wave propagation velocity. As calculated from [40], the effects of the various parameters, film thickness, phase change and interfacial shear, on the wave propagation velocities corresponding to the most amplified wave are presented in figure 13. Almost no influence of the phase change parameter is observed. Furthermore, a general trend of decreasing wave velocity with η^* and τ_i^* is indicated. The major reduction occurs in the range of $\eta^* < 6$. For a sufficiently large film thickness parameter the wave velocity approaches the surface velocity u_i, which in view of [35] can be expressed in dimensionless form by

$$u_i^* = \frac{3}{2} + \frac{1}{4}\frac{\tau_i^* \eta^{*2}}{\mathrm{Re}}. \qquad [55]$$

The asymptotic value of $\frac{3}{2}$ has been already reported for vertical wavy flow free of interfacial shear. For flows with large τ_i^* the wave velocity approaches the surface velocity at much lower film thickness parameters than for the case of small τ_i^*.

Since the disturbance having the maximum growth factor dominates the interface, one should expect fairly well-defined wave velocities corresponding to the most rapidly growing wave. Thus in figure 14 are plotted the wave velocities measured by Jones & Whitaker (1961), Stainthrop & Allen (1965) and Massot & Irani (1966) for isothermal flow of water on a vertical surface. These are compared with the wave velocity of the most amplified wave predicted by [40]. Although the data are too scattered to check the detailed shape of the wave propagation velocity curve, the present calculation at least predicts the observed magnitude and behavior of c_r^* throughout the entire film thickness parameter range investigated by the above authors.

SUMMARY AND CONCLUSION

Area-averaged two-fluid model formulation of a separated two-phase flow system is used to develop a linearized interfacial stability theory. The analysis takes into account the effects of phase change at the interface as well as the dynamic effects of the adjacent vapor flow on the interfacial stability. Instability and wave formation criteria at the interface are established in terms of the generalized parameters. Therefore, one can use

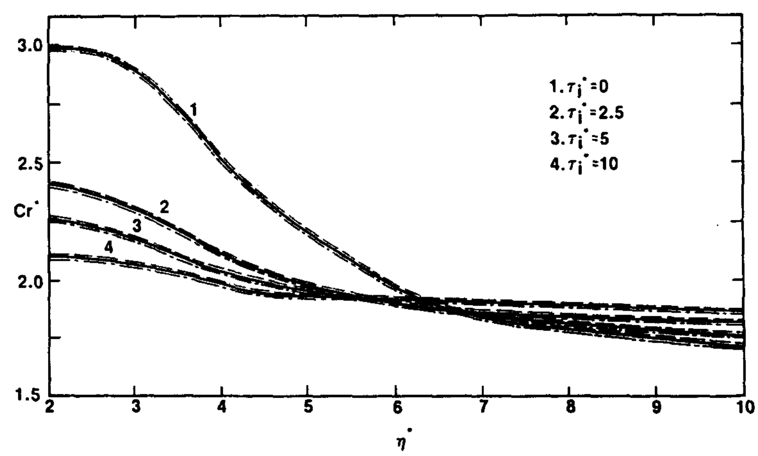

Figure 13. Effects of various parameters on wave velocity of the most unstable wave. —— isothermal, Ku = 0.0; – – –, condensation, Ku = 0.01; – - –, evaporation, Ku = 0.01.

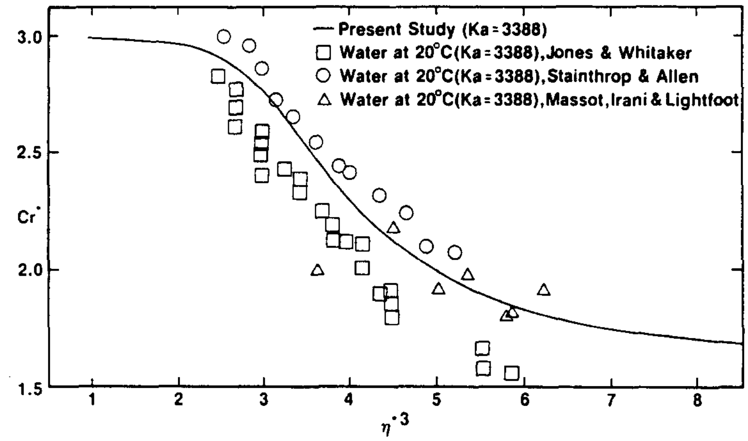

Figure 14. Wave velocity of the most amplified wave as predicted from the present analysis compared with wave velocity data measured by Jones & Whitaker, Stainthrop & Allen, and Massot, Irani & Lightfoot, vertical flow of liquid film (Ku = 0, $g^* = 0$, $\tau_i^* = 0$).

them for a particular separated two-phase flow configuration provided the parameters are calculated properly.

The general criteria are applied to investigate the stability of laminar liquid film flow with interfacial shear and phase change. The influences of the various dimensionless parameters, η^*, Ku, g^* and τ_i^*, on the stability of liquid films are studied with respect to the neutral stability, temporal growth factor and the wave propagation velocity. A general trend of decreasing stability with increasing η^*, g^*, and τ_i^* is observed. The relative degree of the destabilizing effects of these parameters is established by parametric study with reference to water film flow on inclined planes. It is noted that the influences of both gravity and interfacial shear parameters on the stability are more important with a low film thickness parameter. On the other hand, the phase change parameter Ku promotes instability in evaporating liquid films, and stabilizes in condensate film flow. In fact, condensation interfacial phase change induces an absolute stability. When it is possible the theoretical results of the present study are compared with those of other authors. The agreement between both the experiments and the much more complicated calculations based on other methods and the present results is thought to be quite good.

In conclusion, the interfacial stability analysis developed within the frame of a two-fluid model formulation of a separated two-phase flow proves to be quite accurate as judged by comparing its results with the available experimental data and with those of much longer and more complex analytical investigations valid only for the liquid film flow free of interfacial shear. The analysis should be useful in the investigation of the stability of other configurations, such as the Kelvin–Helmholtz and the Rayleigh–Taylor instabilities.

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APPENDIX A

Order of magnitude analysis

The physical system considered is illustrated in figure 15. A liquid film (phase 1) flows on a constant temperature (T_w) under the influence of an adjacent vapor (phase 2) and/or gravity. Both phases are in two-dimensional laminar motion. It is assumed that the vapor is at saturation temperature (T_s) and that the fluid is Newtonian, and that the variations of its thermophysical properties are neglected, except for the density in the momentum equation (Boussinesq approximation). In the following analysis we perform an order of magnitude analysis of the equations of motion to simplify the equations.

The differential equations governing the kinematic and dynamic field in each phase are given as

$$\nabla \cdot \mathbf{v} = 0 \qquad [A1]$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot \mathbf{v}\mathbf{v}\right) = -\nabla P + \nabla \cdot \bar{\bar{\tau}} + \rho \mathbf{g} - \rho\beta(T - T_s)\mathbf{g} \qquad [A2]$$

As noted above the vapor is at saturation temperature; therefore, the last term in [A2] should not appear in the vapor motion equation.

The interfacial mass and momentum balance equations, respectively, are given by Delhaye (1974) as

$$\dot{m}_1 + \dot{m}_2 = 0 \qquad [A3]$$

$$\dot{m}_1(\mathbf{v}_1 - \mathbf{v}_2) + (P_1 - P_2)\hat{n}_1 - \hat{n}_1\cdot(\bar{\bar{\tau}}_1 - \bar{\bar{\tau}}_2) = (\nabla_s\cdot\hat{\eta}_1)\sigma\hat{n}_1 - \nabla_s\sigma \qquad [A4]$$

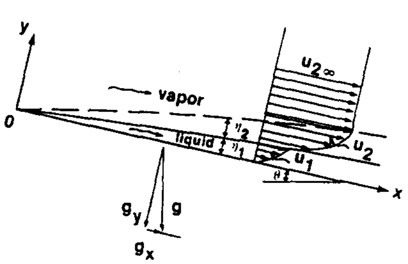

Figure 15. Plane film flow under the influence of adjacent vapor flow and/or gravity.

and the interfacial no relative velocity requirement is expressed by

$$(\mathbf{v}_2 - \mathbf{v}_1)\cdot\hat{t} = 0 \tag{A5}$$

In these equations, ∇_s, $\dot{m}$, $\hat{n}$, and $\hat{t}$ are the divergence operator along the interface, interfacial mass transfer per unit area per unit time, normal unit vector and tangential unit vector, respectively. For a two-dimensional flow field, they are given as follows:

$$\nabla_s = \left(\hat{i} + \frac{\partial\eta_1}{\partial x}\hat{j}\right)\left[1 + \left(\frac{\partial\eta_1}{\partial x}\right)^2\right]^{-1}\frac{\partial}{\partial x} \tag{A6}$$

$$\dot{m}_k = \rho_k(\mathbf{v}_k - \mathbf{v}_i)\cdot\hat{n}_k; \quad k = 1,2 \tag{A7}$$

$$\hat{n}_1 = -\hat{n}_2 = \left(-\frac{\partial\eta_1}{\partial x}\hat{i} + \hat{j}\right)\left[1 + \left(\frac{\partial\eta_1}{\partial x}\right)^2\right]^{-1/2} \tag{A8}$$

$$t = \left(\hat{i} + \frac{\partial\eta_1}{\partial x}\hat{j}\right)\left[1 + \left(\frac{\partial\eta_1}{\partial x}\right)^2\right]^{-1/2} \tag{A9}$$

The field equations together with the interfacial conditions can be simplified by making the thin film approximation in the vapor. The thin film theory is founded on the assumption that the film thickness is small compared with the lateral dimensions of the bounding surface. Thus if η_{1r} denotes a typical value of the film thickness $\eta_1(x,t)$, and l denotes a typical longitudinal dimension, we introduce a dimensionless scaling parameter e_1 by

$$e_1 \equiv \eta_{1r}/l \ll 1. \tag{A10}$$

The thin film approximation given above is purely a geometrical one. However, its dynamic implication for viscous fluids can be obtained by comparing the viscous diffusion time with the characteristic residence time. If u_{1r} is a typical reference velocity in the longitudinal direction, the time a fluid particle spends near the body, the residence time, is approximately l/u_{1r}, while the time required for viscous effects to spread across the film thickness is of order η_{1r}^2/ν_1, where ν_1 is the kinematic viscosity. Then the viscous effects can spread across the film if the diffusion time is shorter than the residence time:

$$\eta_{1r}^2/\nu_1 < l/u_{1r}. \tag{A11}$$

Hence,

$$e_1\,\mathrm{Re}_1 < 1 \tag{A12}$$

where Re_1 is the Reynolds number based on the film thickness, $\mathrm{Re}_1 \equiv \eta_{1r}u_{1r}/\nu_1$.

The orders of magnitude introduced by [A10] and [A12] are in agreement with the boundary layer approximation. Therefore, they can be extended to the vapor flow where a characteristic boundary layer thickness is assumed to be η_{2r}. Thus, it will be assumed for both phases that

$$e_k \equiv \eta_{kr}/l \ll 1, \quad k = 1,2 \tag{A13}$$

and

$$e_k \, \mathrm{Re}_k = <1, \quad k = 1, 2. \tag{A14}$$

We can compare the magnitudes of the various terms in [A1]–[A5]. For this purpose, the following dimensionless variables, all of which are sure *to be order of unity*, are introduced:

$$\begin{aligned} &x^+ \equiv x/l; \quad y_k^+ \equiv y_k/\eta_{kr} \\ &u_k^+ \equiv u_k/u_{kr}; \quad v_k^+ \equiv v_k/e_k u_{kr} \\ &P_k^+ \equiv (P_k - P_r)/(\rho_k u_{kr}^2); \quad t^+ \equiv t\omega_r \\ &\dot{m}_1^+ \equiv \dot{m}_1/(e_1\rho_1 u_{1r}); \quad T^+ \equiv (T_1 - T_s)/(T_w - T_s) \\ &\Delta u^+ \equiv \Delta u/(e_1^2 u_{1r}\Delta\rho/\rho_2); \quad \Delta v^+ \equiv \Delta v/(e_1 u_{1r}\Delta\rho/\rho_2) \end{aligned} \tag{A15}$$

where ω_r is a typical frequency of the interfacial motion, whereas Δu and Δv are phase 2 to phase 1 longitudinal and transversal velocity differences at the interface.

Before introducing these dimensionless parameters into [A1]–[A5], several observations can be made. First, in order to be sure u_k^+ is order of unity, a typical value of interfacial velocity should be used for the liquid ($u_{1r} \equiv u_{ir}$), whereas $u_{2r} \equiv u_{2\infty} - u_{ir}$ can be used for the vapor motion. Second, it may be pointed out that the pressure drop is nondimensionalized so as to render the pressure gradient and the inertia effects of equal magnitude, a fact that is well known in the boundary layer analysis. Furthermore, noting that the pressure drops in both phases are at the same order it can be seen that $\rho_1 u_{1r}^2 \simeq \rho_2 u_{2r}^2$.

When the parameters defined by [A15] are introduced in [A1]–[A5] we obtain the following dimensionless equations:

Continuity equation

$$\frac{\partial u^+}{\partial x^+} + \frac{\partial v^+}{\partial y^+} = 0. \tag{A16}$$

x Component of momentum equation

$$\begin{aligned} e\,\mathrm{Re}\left(S\frac{\partial u^+}{\partial t^+} + u^+\frac{\partial u^+}{\partial x^+} + v^+\frac{\partial u^+}{\partial y^+}\right) = &-e\,\mathrm{Re}\frac{\partial p^+}{\partial x^+} + (\mathrm{Re}/\mathrm{Fr})g_x^+ \\ &- (\beta\Delta T)(\mathrm{Re}/\mathrm{Fr})g_x^+T^+ + \left(e^2\frac{\partial^2 u^+}{\partial x^{+2}} + \frac{\partial^2 u^+}{\partial y^{+2}}\right). \end{aligned} \tag{A17}$$

y Component of momentum equation

$$\begin{aligned} e^2\,\mathrm{Re}\left(S\frac{\partial v^+}{\partial t^+} + u^+\frac{\partial v^+}{\partial x^+} + v^+\frac{\partial v^+}{\partial y^+}\right) = &-\mathrm{Re}\frac{\partial p^+}{\partial y^+} + (\mathrm{Re}/\mathrm{Fr})g_x^+ - (\beta\Delta T)(\mathrm{Re}/\mathrm{Fr})g_y^+T^+ \\ &+ e\left(e^2\frac{\partial^2 v^+}{\partial x^{+2}} + \frac{\partial^2 v^+}{\partial y^{+2}}\right). \end{aligned} \tag{A18}$$

Interfacial mass balance

$$\dot{m}_1^+ = \Delta v^+\left[1 + e_1^2\left(\frac{\partial\eta_1^+}{\partial x^+}\right)^2\right]^{1/2} = \left(-u_1^+\frac{\partial\eta_1^+}{\partial x^+} + v_1^+ - S\frac{\partial\eta_1^+}{\partial t^+}\right)\cdot\left[1 + e_1^2\left(\frac{\partial\eta_1^+}{\partial x^+}\right)^2\right]^{-1/2}. \tag{A19}$$

Normal component of interfacial momentum balance

$$
\begin{aligned}
&-\left(\frac{\Delta\rho}{\rho_2}\right)e_1^2\,\mathrm{Re}_1\,\dot{m}^{+2}+(P_1^+-P_2^+)\,\mathrm{Re}_1-2e_1\left\{\left[\frac{\partial u_1^+}{\partial x^+}-\left(\frac{\mu_2}{\mu_1}\right)\left(\frac{\rho_1}{\rho_2}\right)^{1/2}\frac{\partial u_2^+}{\partial x^+}\right]\left(\frac{\partial u_1^+}{\partial x^+}\right)^2\right.\\
&\quad-\left[\left(\frac{\partial u_1^+}{\partial y_1^+}+e_1^2\frac{\partial v_1^+}{\partial x^+}\right)-\left(\frac{\mu_2}{\mu_1}\right)\left(\frac{\rho_1}{\rho_2}\right)^{1/2}\left(\frac{e_1}{e_2}\right)\left(\frac{\partial u_2^+}{\partial y_2^+}+e_2^2\frac{\partial v_2^+}{\partial x^+}\right)\right]\left(\frac{\partial \eta_1^+}{\partial x^+}\right)\\
&\quad\left.+\left[\frac{\partial v_1^+}{\partial y_1^+}-\left(\frac{\mu_2}{\mu_1}\right)\left(\frac{\rho_1}{\rho_2}\right)^{1/2}\left(\frac{e_1}{e_2}\right)\frac{\partial v_2^+}{\partial y_2^+}\right]\right\}\left[1+e_1^2\left(\frac{\partial \eta_1^+}{\partial x^+}\right)^2\right]^{-1}\\
&=-e_1^2\,\mathrm{We}\,\mathrm{Re}_1\left(\frac{\partial^2\eta_1^+}{\partial x^{+2}}\right)\cdot\left[1+e_1^2\left(\frac{\partial \eta_1^+}{\partial x^+}\right)^2\right]^{-3/2}
\end{aligned}
$$

Tangential component of interfacial momentum balance

$$
\begin{aligned}
&2e_1^2\left(\frac{\partial \eta_1^+}{\partial x^+}\right)\left\{\left[\frac{\partial v_1^+}{\partial y_1}-\left(\frac{\mu_2}{\mu_1}\right)\left(\frac{\rho_1}{\rho_2}\right)^{1/2}\frac{\partial v_2^+}{\partial y_2^+}\right]-\left[\frac{\partial u_1^+}{\partial x^+}-\left(\frac{\mu_2}{\mu_1}\right)\left(\frac{\rho_1}{\rho_2}\right)^{1/2}\frac{\partial u_2^+}{\partial u^+}\right]\right\}\\
&\quad+\left[1-e_1^2\left(\frac{\partial \eta_1^+}{\partial x^+}\right)^2\right]\left[\left(\frac{\partial u_1^+}{\partial y^+}+e_1^2\frac{\partial v_1^+}{\partial x^+}\right)-\left(\frac{\mu_2}{\mu_1}\right)\left(\frac{\rho_1}{\rho_2}\right)^{1/2}\left(\frac{e_1}{e_2}\right)\left(\frac{\partial u_2^+}{\partial y^+}+e_2^2\frac{\partial v_2^+}{\partial x^+}\right)\right]\\
&=e_1\,\mathrm{Ma}\frac{\partial T_s^+}{\partial x^+}\left[1+e_1^2\left(\frac{\partial \eta_1^+}{\partial x^+}\right)^2\right]^{1/2} \qquad \text{[A21]}
\end{aligned}
$$

where Fr, S, We and Ma are respectively the Froude, Strouhal, Weber and Marangoni numbers defined as:

$$
\begin{aligned}
&\mathrm{Fr}^2 \equiv u_r/\eta_r g; \quad \mathrm{S} \equiv \omega_r l/u_{1r}\\
&\mathrm{We} \equiv \sigma/\eta_{1r}\rho_1 u_{1r}^2; \quad \mathrm{Ma} \equiv C_\sigma \Delta T/\mu_1 u_{1r}. \qquad \text{[A22]}
\end{aligned}
$$

C_σ being the surface tension gradient with respect to temperature. It is a well-known fact that surface tension gradients may be caused by gradients in surface temperature, by gradients in the concentration of a solute, and by gradients in electrical potential if there is an electric charge at the interface. In the analysis presented here, only the former one is considered, and the surface tension gradient is evaluated by noting that the vapor is at saturation state. Thus,

$$\frac{\partial \sigma}{\partial x}=\left(\frac{\mathrm{d}\sigma}{\mathrm{d}T_2}\right)\left(\frac{\mathrm{d}T_2}{\mathrm{d}P_2}\right)\frac{\partial P_2}{\partial x} \qquad \text{[A23]}$$

where $(\mathrm{d}T_2/\mathrm{d}P_2)$ is evaluated through the use of the Clausius–Clapeyron relation.

The magnitude of the various terms appearing in [A16]–[A21] can be assessed in view of the scales introduced by [A13] and [A14]. Terms with a value of (e Re) and higher are retained. The lower are discarded. After the order of magnitude analysis is performed, these equations are rewritten in the more common dimensional form as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \text{[A24]}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial P}{\partial x}+\rho g_x+\mu_1\frac{\partial^2 u}{\partial y^2} \qquad \text{[A25]}$$

$$0 = -\frac{\partial P}{\partial y} + \rho g_y \qquad \text{[A26]}$$

$$\dot{m}_1 = (v_{2i} - v_{1i})\rho_1\rho_2/\Delta\rho = \rho_1\left(v_{1i} - u_{1i}\frac{\partial \eta_1}{\partial x} - \frac{\partial \eta_1}{\partial t}\right) \qquad \text{[A27]}$$

$$P_{1i} - P_{2i} = \dot{m}_1^2(\Delta\rho/\rho_1\rho_2) - \sigma\frac{\delta^2\eta_1}{\partial x^2} \qquad \text{[A28]}$$

$$\mu_1\left(\frac{\partial u_1}{\partial y}\right)_i - \mu_2\left(\frac{\partial u_2}{\partial y}\right)_i = 0. \qquad \text{[A29]}$$

In arriving at the above set of equations, a comparison of various terms appearing in [A17]–[A21] was carried out for the case of $e_k \ll 1$ and $e_k\mathrm{Re}_k < 1$, and the following considerations have been taken into account with respect to the order of dimensionless groups such as Re/Fr, $\beta\Delta T$, $(\mu_2/\mu_1)(\rho_1/\rho_2)^{1/2}$, Ma and $\Delta\rho/\rho_2$:

(1) Four classes of fluids, namely, common fluids, refrigerants, liquid metals and hydrocarbons, have been considered. For the purpose of using proper thermophysical properties, water for common fluids, Freon-12 for refrigerants, sodium for liquid metals, and, finally, methane for hydrocarbons have been selected as representative fluids.

(2) It has been assumed that (e Re) is about order of 10^{-2}.

(3) Considering the free film flow, the ratio of (Re/Fr) is order of unity. Since the value of β for the fluids listed above can only be order $10^{-4} - 10^{-3}$ over a wide range of system pressure, and since $(T_w - T_s)$ can only be a few degrees to prevent the nucleate boiling which is accompanied by a lower temperature difference in thin liquid film than in pool boiling, the free convection term appearing in momentum equations has been neglected. That is to say that the buoyancy-driven convection is not the dominating mode of instability.

(4) When evaluated for a wide range of operating conditions, it was observed that the values of $(\mu_2/\mu_1)(\rho_1/\rho_2)^{1/2}$ are less than 4 for water, Freon-12 and sodium, and far less than unity for methane. Therefore, this term appearing in the interfacial momentum balance equations has been assumed to be order of unity.

(5) The Marangoni effect which appears in the tangential component of the interfacial momentum balance has been neglected because the value of $(C_\sigma T_s\Delta\rho/\rho_1\rho_2 h_{LG})$ changes between 10^{-9} and 10^{-7} m for the fluids listed above. As pointed out by Sreenivasan & Lin (1978) this effect can be relevant only to a very thin film flow in which neither the gravity-capillary ripples nor the buoyancy-driven convection is the dominating mode of instability.

(6) In spite of the fact that $e_1 \ll 1$, the term $(\Delta\rho/\rho_2)e_1^{\,2}\,\mathrm{Re}_1\, m_1^{+2}$ has been kept in the normal component of interfacial momentum balance because $(\Delta\rho/\rho_2)$ can be very large for very low system pressure operations. It is probably not as important for normal conditions.

The stability analysis presented in the main body of this manuscript is based on the integral equations obtained from the simplified equations, [A24]–[A29].

APPENDIX B

Coefficients a's which appear in [20] are defined as follows:†

$$a_1 = \left(\frac{A}{\xi_i}\right) \qquad \text{[B1]}$$

†For the purpose of simplicity bar over these flow variables was dropped. It is understood that all quantities which appear in these equations belong to the base flow.

$$a_2 = \Delta\rho g_y\left(\frac{A}{\xi_i}\right) - \left[\frac{\rho_1}{1-\alpha}\operatorname{Cov}(u_1^2) + \frac{\rho_2}{\alpha}\operatorname{Cov}(u_2^2)\right] + \frac{\partial}{\partial\alpha}[\rho_1\operatorname{Cov}(u_1^2) - \rho_2\operatorname{Cov}(u_2^2)] + \frac{\Delta\rho}{\rho_1\rho_2}\dot{m}_{1i}\frac{\partial\dot{m}_{1i}}{\partial\alpha} \quad \text{[B2]}$$

$$a_3 = -\frac{\partial}{\partial\langle u_2\rangle}[\rho_1\operatorname{Cov}(u_1^2) - \rho_2\operatorname{Cov}(u_2^2)] \quad \text{[B3]}$$

$$a_4 = \frac{\partial}{\partial\langle u_1\rangle}[\rho_1\operatorname{Cov}(u_1^2) - \rho_2\operatorname{Cov}(u_2^2)] \quad \text{[B4]}$$

$$\begin{aligned} a_5 = {} & \left[\frac{u_i-\langle u_1\rangle}{(1-\alpha)^2} - \frac{u_i-\langle u_2\rangle}{\alpha^2}\right]\dot{m}_{1i}\left(\frac{\xi_i}{A}\right) - \left[\frac{1}{(1-\alpha)^2} - \frac{1}{\alpha}\right]\tau_{1i}\left(\frac{\xi_i}{A}\right) + \frac{\tau_1 e}{(1-\alpha)^2}\left(\frac{\xi_{1e}}{A}\right) \\ & + \frac{\tau_{2e}}{\alpha}\left(\frac{\xi_{2e}}{A}\right) + \left[\frac{1}{1-\alpha}\frac{\partial}{\partial\alpha}(u_i - \langle u_1\rangle) + \frac{1}{\alpha}\frac{\partial}{\partial\alpha}(u_i-\langle u_2\rangle)\right]\dot{m}_{1i}\left(\frac{\xi_i}{A}\right) \\ & + \left(\frac{u_i-\langle u_1\rangle}{1-\alpha} + \frac{u_i-\langle u_2\rangle}{\alpha}\right)\left(\frac{\partial\dot{m}_{1i}}{\partial\alpha}\right)\left(\frac{\xi_i}{A}\right) - \left(\frac{1}{1-\alpha}+\frac{1}{\alpha}\right)\left(\frac{\partial\tau_{1i}}{\partial\alpha}\right)\left(\frac{\xi_i}{A}\right) \\ & + \left(\frac{1}{1-\alpha}\right)\left(\frac{\partial\tau_{1e}}{\partial\alpha}\right)\left(\frac{\xi_{1e}}{A}\right) - \frac{1}{\alpha}\left(\frac{\partial\tau_{2e}}{\partial\alpha}\right)\left(\frac{\xi_{2e}}{A}\right) \end{aligned} \quad \text{[B5]}$$

$$\begin{aligned} a_6 = {} & -\frac{\partial}{\partial\langle u_2\rangle}\left(\frac{u_i-\langle u_1\rangle}{1-\alpha} + \frac{u_i-\langle u_2\rangle}{\alpha}\right)\dot{m}_{1i}\left(\frac{\xi_i}{A}\right) + \left(\frac{1}{1-\alpha}+\frac{1}{\alpha}\right)\left(\frac{\partial\tau_i}{\partial\langle u_2\rangle}\right)\left(\frac{\xi_i}{A}\right) \\ & - \frac{1}{1-\alpha}\left(\frac{\partial\tau_{1e}}{\partial\langle u_2\rangle}\right)\left(\frac{\xi_{1e}}{A}\right) + \frac{1}{\alpha}\left(\frac{\partial\tau_{2e}}{\partial\langle u_2\rangle}\right)\left(\frac{\xi_{2e}}{A}\right) \end{aligned} \quad \text{[B6]}$$

$$\begin{aligned} a_7 = {} & \frac{\partial}{\partial\langle u_1\rangle}\left(\frac{u_i-\langle u_1\rangle}{1-\alpha} + \frac{u_i-\langle u_2\rangle}{\alpha}\right)\dot{m}_{1i}\left(\frac{\xi_i}{A}\right) - \left(\frac{1}{1-\alpha}+\frac{1}{\alpha}\right)\left(\frac{\partial\tau_{1i}}{\partial\langle u_1\rangle}\right)\left(\frac{\xi_i}{A}\right) \\ & + \frac{1}{1-\alpha}\left(\frac{\partial\tau_{1e}}{\partial\langle u_1\rangle}\right)\left(\frac{\xi_{1e}}{A}\right) - \frac{1}{\alpha}\left(\frac{\partial\tau_{2e}}{\partial\langle u_1\rangle}\right)\left(\frac{\xi_{2e}}{A}\right). \end{aligned} \quad \text{[B7]}$$

Coefficients b's which appear in [23] are defined as follows:

$$b_1 = \frac{1}{2}\left(\frac{\rho_1}{1-\alpha}+\frac{\rho_2}{\alpha}\right)^{-1}\left[\left(\frac{a_3}{\alpha}+\frac{a_4}{1-\alpha}\right) + 2\left(\frac{\rho_2\langle u_2\rangle}{\alpha}+\frac{\rho_1\langle u_1\rangle}{1-\alpha}\right)\right] \quad \text{[B8]}$$

$$b_2 = \left(\frac{1}{2k}\right)\left(\frac{\rho_1}{1-\alpha}+\frac{\rho_2}{\alpha}\right)^{-1}\left[-\left(\frac{a_7}{1-\alpha}+\frac{a_6}{\alpha}\right) + \left(\frac{1}{1-\alpha}+\frac{1}{\alpha}\right)\left(\frac{\epsilon k_1\Delta T}{(1-\alpha)^2 h_{LG}}\right)\left(\frac{\xi_i}{A}\right)^2\right] \quad \text{[B9]}$$

$$\begin{aligned} b_3 = {} & \left(\frac{\rho_1}{1-\alpha}+\frac{\rho_2}{\alpha}\right)^{-1}\left\{\sigma\left(\frac{\xi_i}{A}\right)k^2 - a_2 + \left(\frac{a_3\langle u_2\rangle}{\alpha}+\frac{a_4\langle u_1\rangle}{1-\alpha}\right)\right. \\ & \left. + \left(\frac{\rho_2\langle u_2\rangle^2}{\alpha}+\frac{\rho_1\langle u_1\rangle^2}{1-\alpha}\right)\right] - \left(\frac{a_6}{\rho_2\alpha}+\frac{a_7}{\rho_1(1-\alpha)}\right)\left(\frac{\epsilon k_1\Delta T}{(1-\alpha)^2 h_{LG}}\right)\left(\frac{1}{k^2}\right)\left(\frac{\xi_i}{A}\right)^2\right\} \end{aligned} \quad \text{[B10]}$$

$$\begin{aligned} b_4 = {} & \left(\frac{\rho_1}{1-\alpha}+\frac{\rho_2}{\alpha}\right)^{-1}\left(\frac{1}{k}\right)\left\{a_5 + \left(\frac{a_6\langle u_2\rangle}{\alpha}+\frac{a_7\langle u_1\rangle}{1-\alpha}\right) - \left[\left(\frac{a_3}{\rho_2\alpha}+\frac{a_4}{\rho_1(1-\alpha)}\right)\right.\right. \\ & \left.\left. + \left(\frac{\langle u_1\rangle}{1-\alpha}+\frac{\langle u_2\rangle}{\alpha}\right)\right]\left(\frac{\epsilon k_1\Delta T}{(1-\alpha)^2 h_{LG}}\right)\left(\frac{\xi_i}{A}\right)^2\right\} \end{aligned} \quad \text{[B11]}$$